THE

MATHEMATICAL GAZETTE.

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SOME USEFUL GROUPS IN THE TEACHING OF ELEMENTARY TRIGONOMETRY.

THE operation (c) of taking the complement of an angle (α) is of order (period) two, since $90°-(90°-\alpha)=\alpha$. Similarly, the operation (s) of taking the supplement of an angle is of order two. If we perform these two operations in succession there results an operation (cs), which is of order four since it changes α into $\alpha+90°$. Hence the group $\{c, s\}$ generated by c and s is the dihedral rotation group of order 8. This fact follows directly from the general theorem, *Two operations of order two generate the dihedral rotation group whose order is twice the order of their product.** For the sake of simplicity we give the following independent proof of this fact:

Using rectangular co-ordinates and measuring angles in the ordinary manner, it is evident that the operation c and s are equivalent to the reflections of the terminal line of α on the bisector of the first quadrant and the y-axis respectively. These operations transform into itself the square whose centre is at the origin and whose sides are parallel to the axes. Hence this square is transformed into itself by all the operations which may be obtained by combining c and s in every possible way. That is, it is transformed into itself by $\{c, s\}$. Since it is known (and can readily be verified) that the group of movements of the square is the dihedral rotation group of order 8, it follows that $\{c, s\}$ is contained in this group. It could not be of order 4 since c and s are non-commutative. As the order of a sub-group divides the order of the group, $\{c, s\}$ must be identical with the group of movements of the square.

* *Bulletin of the American Mathematical Society*, vol. 7 (1901), p. 424.

This group is of great historic interest since it is the one which enables us to reduce the solution of the general equation of the fourth degree to that of a cubic. In trigonometry it is perhaps most prominent as the complement-supplement group, while in arithmetic it appears as the group of the operations of subtracting from 2 and dividing by 2.* In elementary geometry it appears as the group of movements of the square, as was noted above. It is commonly called the octic group even if it does not have much greater claims for usefulness than several other groups of the same order—especially the quaternion group. We proceed to consider some fundamental uses of the octic group in elementary trigonometry.

The operations of this group may be written as follows:

$$1 \quad cs \quad cscs \quad sc$$
$$c \quad s \quad scs \quad csc.$$

If these operations are applied to α the corresponding angles are

$$\alpha \qquad \alpha+90 \qquad \alpha+180^\circ \quad \alpha-90$$
$$90^\circ-\alpha \quad 180^\circ-\alpha \quad 270^\circ-\alpha \qquad -\alpha.$$

It is very interesting to observe that these are the eight angles whose trigonometric functions are expressed in terms of functions of α in the early part of the text-books on elementary trigonometry. The trigonometric functions of all of these angles can be directly obtained from those of any two which correspond to generators of the octic group. For instance, if we know the trigonometric functions of the complement and the supplement of an angle, those of the other angles can be directly obtained by means of the given group symbols. We shall do this for sine and cosine†

$$\sin(\alpha+90^\circ)=\sin csa=\sin ca=\cos\alpha$$
$$\cos(\alpha+90^\circ)=\cos csa=-\cos ca=-\sin\alpha$$
$$\sin(\alpha+180^\circ)=\sin cs\,.\,csa=\cos csa=-\sin\alpha$$
$$\cos(\alpha+180^\circ)=\cos cs\,.\,csa=-\sin csa=-\cos\alpha$$
$$\sin(\alpha-90^\circ)=\sin sca=\cos sa=-\cos\alpha$$
$$\cos(\alpha-90^\circ)=\cos sca=\sin sa=\sin\alpha$$
$$\sin(270^\circ-\alpha)=\sin sc\,.\,sa=\sin sca=-\cos\alpha$$
$$\cos(270^\circ-\alpha)=\cos sc\,.\,sa=-\cos sca=-\sin\alpha$$
$$\sin(-\alpha)=\sin cs\,.\,ca=\cos csa=-\sin\alpha$$
$$\cos(-\alpha)=\cos cs\,.\,ca=\sin csa=\cos\alpha.$$

If we represent the operation of increasing an angle by 90° by p and suppose the functions of $\alpha+90^\circ$ as well as those of

* *Quarterly Journal of Mathematics*, vol. 37 (1905), p. 82.

† Cf. *Quarterly Journal of Mathematics*, vol 37, (1906), p. 228.

$90°-\alpha$ to be known, we may readily find those of the others in the following manner:

$$\sin(\alpha+180°)=\sin p^2\alpha=\cos p\alpha=-\sin\alpha$$
$$\cos(\alpha+180°)=\cos p^2\alpha=-\sin p\alpha=-\cos\alpha$$
$$\sin(\alpha-90°)=\sin p^3\alpha=-\sin p\alpha=-\cos\alpha$$
$$\cos(\alpha-90°)=\cos p^3\alpha=-\cos p\alpha=\sin\alpha$$
$$\sin(180°-\alpha)=\sin cp\alpha=\cos c\alpha=\sin\alpha$$
$$\cos(180°-\alpha)=\cos cp\alpha=-\sin c\alpha=-\cos\alpha$$
$$\sin(270°-\alpha)=\sin cp^2\alpha=-\sin c\alpha=-\cos\alpha$$
$$\cos(270°-\alpha)=\cos cp^2\alpha=-\cos c\alpha=-\sin\alpha$$
$$\sin(-\alpha)=\sin cp^3\alpha=-\cos c\alpha=-\sin\alpha$$
$$\cos(-\alpha)=\cos cp^3\alpha=\sin c\alpha=\cos\alpha.$$

The advantages of the present point of view result from the fact that it associates the group and these eight angles in such a way that a knowledge of the one throws light on the other. If we know the octic group, we know directly what two angles may be made fundamental in the study of the others.* The necessary and sufficient condition is that the two angles correspond to generators of the octic group. In other words, they must correspond to two non-commutative operations of this group, since any two non-commutative operations of a non-abelian group of order p^3, p being a prime number, generate the group and any two generators are non-commutative.

The properties of the octic group suggest a large number of exercises with respect to the eight angles under consideration. For instance, the functions of all of them may be expressed in terms of the functions of any other one of them just as easily as in terms of the functions of α. Since the octic group has only one invariant operation besides the identity the corresponding angle cannot be used as one of the two fundamental angles, while each of the non-invariant operations can be used as one of these two angles. If α is the angle corresponding to the identity, as above, $\alpha+180°$ corresponds to the invariant operation of order 2.

The use of the octic group makes the work entirely analytic after the trigonometric functions of the fundamental angles are known. The main interest, however, lies in the connections which it exhibits and the comprehensive view which it affords. The teacher should be familiar with this point of view even if he does not consider it wise to present it to the beginner.

Instead of considering the operations of subtracting from 90° and 180° we might consider those of subtracting from any other

*We could not use only one fundamental angle since the octic group is non-cyclic.

angles.* For instance, if t represents the operation of subtracting from 45° while c has the same meaning as before, the operation tc is of order 8, and hence $\{c, t\}$ is the dihedral rotation group of order 16. The operations of this group and the corresponding angles may be written in the same manner as those of the octic group. In fact, it is easy to see that $\{c, t\}$ includes $\{c, s\}$, so that the results obtained above will be appear as a special case under this larger group. We proceed to give a few illustrative examples.

$$\sin(\alpha+45^\circ)=\sin tc\alpha=\cos t\alpha=\tfrac{1}{2}\sqrt{2}(\cos\alpha+\sin\alpha)\dagger$$
$$\cos(\alpha+45^\circ)=\cos tc\alpha=\sin t\alpha=\tfrac{1}{2}\sqrt{2}(\cos\alpha-\sin\alpha)$$

$$\sin(\alpha+90^\circ)=\sin(tc)^2\alpha=\tfrac{1}{2}\sqrt{2}(\cos tc\alpha+\sin tc\alpha)$$
$$=\tfrac{1}{2}(\cos\alpha-\sin\alpha+\cos\alpha+\sin\alpha)=\cos\alpha$$

$$\cos(\alpha+90^\circ)=\cos(tc)^2\alpha=\tfrac{1}{2}\sqrt{2}(\cos tc\alpha-\sin tc\alpha)$$
$$=\tfrac{1}{2}(\cos\alpha-\sin\alpha-\cos\alpha-\sin\alpha)=-\sin\alpha$$

$$\sin(\alpha+180^\circ)=\sin(tc)^4\alpha=\cos(tc)^2\alpha=-\sin\alpha$$
$$\cos(\alpha+180^\circ)=\cos(tc)^4\alpha=-\sin(tc)^2\alpha=-\cos\alpha$$

$$\sin(\alpha-90^\circ)=\sin(tc)^6\alpha=-\sin(tc)^2\alpha=-\cos\alpha$$
$$\cos(\alpha-90^\circ)=\cos(tc)^6\alpha=-\cos(tc)^2\alpha=\sin\alpha$$

$$\sin(-\alpha)=\sin(tc)^2c\alpha=\cos(tc)^2\alpha=-\sin\alpha$$
$$\cos(-\alpha)=\cos(tc)^2c\alpha=\sin(tc)^2\alpha=\cos\alpha$$

$$\sin(\alpha+135^\circ)=\sin(tc)^3\alpha=\cos tc\alpha=\tfrac{1}{2}\sqrt{2}(\cos\alpha-\sin\alpha)$$
$$\cos(\alpha+135^\circ)=\cos(tc)^3\alpha=-\sin tc\alpha=-\tfrac{1}{2}\sqrt{2}(\cos\alpha+\sin\alpha).$$

These examples seem sufficient to illustrate the method of work. The general theory of these subtraction groups is given in the articles published in the *Annals of Mathematics* to which reference is made above. The object of the method is to exhibit relations which are apt to be overlooked. It aims to clarify and intensify the knowledge of fundamental facts rather than to extend this knowledge. Some of the relations between the functions of the same angle afford another very interesting application of the group concept. The two operations of subtracting from unity and dividing unity may be represented by s_1, s_2 respectively. It is evident that each of these operations is of order 2 and that their product s_1s_2 is of order 3. Hence they generate the dihedral rotation group of order 6. This is well illustrated by the six values of an anharmonic ratio and the group of movements of the equilateral triangle.

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* Cf. *Annals of Mathematics*, vol. 6 (1905), p. 41.

† Since t is one of the assumed generators of our group of order 16 the functions of the angle $(45^\circ-\alpha)$ which corresponds to t are supposed to be known.

The six operations of $\{s_1, s_2\}$ may be represented as follows:

$$\begin{array}{lll} 1 & s_1s_2 & s_2s_1 \\ s_1 & s_2 & s_1s_2s_1. \end{array}$$

If we perform these operations on $\sin^2\alpha$ the resulting functions are, in order,

$$\begin{array}{llll} \sin^2\alpha & \sec^2\alpha & -\cot^2\alpha \\ \cos^2\alpha & \csc^2\alpha & -\tan^2\alpha. \end{array}$$

Since the six operations in question constitute a group the same six functions are obtained if the operations are performed on any one of them. This method exhibits not only the close relations between these functions and the values of an anharmonic ratio, but it may also be regarded as a special case of a group of subtraction and division. The fundamental properties of these groups have been developed in an article published in volume 37 of the *Quarterly Journal of Mathematics*. The same volume contains other interesting examples of the application of group theory to trigonometry.

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ON HIGHER TRIGONOMETRY.

III. THE ELEMENTARY TRANSCENDENTAL FUNCTIONS.

Algebraic Lemma.

The following system of algebraic inequalities is of great value and of wide application:

$$\text{(i)}\quad 1-\alpha<\frac{1}{1+\alpha},\quad 1+\alpha<\frac{1}{1-\alpha}$$

for all positive proper fractional values of α;

$$\text{(ii)}\quad \prod_{r=1}^{r=n}(1-\alpha_r)>1-\Sigma_1+\Sigma_2-\ldots-\Sigma_{2k-1}$$

$$\text{and}\quad <1-\Sigma_1+\Sigma_2-\ldots+\Sigma_{2k}$$

for positive proper fractional values of $\alpha_1, \alpha_2, \ldots \alpha_n$, and values of k such that $2k<n$; Σ_r denoting the sum of the r-products of the α's.

These inequalities (ii) may be given otherwise in the form

$$\Pi(1-\alpha)=1-\Sigma_1+\Sigma_2\ldots+(-)^{k-1}.\Sigma_{k-1}+(-)^k.\phi_k.\Sigma_k,$$

where ϕ_k is a function of $\alpha_1, \alpha_2, \ldots \alpha_n$, whose value is a positive proper fraction when positive proper fractional values are given to the α's. In particular

$$1-n\alpha<(1-\alpha)^n<1-n\alpha+\frac{n(n-1)}{1.2}\alpha^2,$$

for positive proper fractional values of a, and positive integral values of n.

The inequalities $1>\Pi(1-a)>1-\Sigma_1$ have been freely used, *e.g.* in the theory of infinite products, but it does not seem to have been noted that they admit of the above useful extension.

The Exponential Function.

The function of z and n which is expressed by $(1+z/n)^n$ can be expanded for all values of z and for positive integral values of n, and this enables us to prove that it tends to a limit (a function of z alone) as n tends to infinity by integral variation. This function of z can be shown to have simple and interesting properties which make it fundamental to the higher analysis, and on account of which it is designated the exponential function:

$$(1+z/n)^n=1+z+p_1\cdot\frac{z^2}{2!}+p_2\cdot\frac{z^3}{3!}+\ldots+p_{n-1}\cdot\frac{z^n}{n!},$$

where p_r denotes the product $(1-1/n)(1-2/n)\ldots(1-r/n)$; it is clear from the Lemma that $1>p_r>1-r(r+1)/2n$ for all values of r considered, and therefore, if we write $p_r=1-\epsilon_r$, ϵ_r tends to zero as n tends to infinity, *for finite values of* r.

Hence, k having a positive integral value, we may write

$$(1+z/n)^n=E(z,k)-\eta(z,k,n)$$

$$+p_{k-1}\cdot\frac{z^k}{k!}\left\{1+\left(1-\frac{k}{n}\right)\frac{z}{k+1}+\left(1-\frac{k}{n}\right)\left(1-\frac{k+1}{n}\right)\frac{z^2}{(k+1)(k+2)}+\ldots\right\}$$

where $E(z,k)\equiv 1+z+\dfrac{z^2}{2!}+\ldots+\dfrac{z^{k-1}}{(k-1)!}$ and is independent of n,

$$\eta(z,k,n)\equiv\epsilon_1\cdot\frac{z^2}{2!}+\epsilon_2\cdot\frac{z^3}{3!}+\ldots+\epsilon_{k-2}\cdot\frac{z^{k-1}}{(k-1)!},$$

and tends to zero when n tends to infinity, *for finite values of* k *and* z.

Thus $|(1+z/n)^n-E(z,k)|<|\eta(z,k,n)|+|z|^k/(k-|z|)\cdot(k-1)!$,

provided that $k>|z|$, by the ordinary method of applying a geometric approximation to obtain an upper limit to the modulus of the remainder.

Now the function $E(z,k)$ can be shown by Limit-Theorem IV. above [again applying the geometric series approximation] to tend to a limit, $E(z)$, as k tends to infinity, for all finite values of z; *this is generally written*

$$E(z)=1+z+\frac{z^2}{2!}+\ldots+\frac{z^r}{r!}+\ldots \text{ ad inf.},$$

the "infinite series" on the right being "convergent" for all finite values of z.*

And we can now write

$$|(1+z/n)^n - E(z)| < |\delta(z, k)| + |\eta(z, k, n)| + \gamma(z, k),$$

where $\delta(z, k)$ is a function which tends to zero as k tends to infinity for all finite values of z; and $\gamma(z, k) = |z|^k/(k-|z|)(k-1)!$

Then, corresponding to any assigned finite positive real number, ϵ, *however small*, we can choose

(i) a value of k so great that $|\delta(z, k)| < \epsilon/3$ and that $\gamma(z, k) < \epsilon/3$.

(ii) a value of n so much greater that $|\eta(z, k, n)| < \epsilon/3$,

i.e. a value (N) of n can be found such that

$$|(1+z/n)^n - E(z)| < \epsilon, \text{ provided } n > N,$$

and therefore $(1+z/n)^n$ *tends to the limit* E(z) *as* n *tends to infinity by integral variation, for all finite values of* z.

[In the particular cases given by the values $+1$ and -1 of z the student may obtain exercise in the new ideas by going over the proof and applying the more obvious Limit-Theorems II. and III.; establishing

(i) that $\mathrm{Lt}(1+1/n)^n$ is equal to the limiting value of the increasing real function $(1+1+1/2!+\ldots+1/n)$;

(ii) that $\mathrm{Lt}(1-1/n)^n$ is equal to the common limiting value of the two real functions

$$\left(\frac{1}{2!} - \frac{1}{3!} + \ldots - \frac{1}{(2n-1)!}\right), \quad \left(\frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{1}{2n!}\right),$$

the former increasing, the latter decreasing. This case arises out of the inequalities connected with $(1-1/n)^n$; the course of higher trigonometry may satisfactorily commence with it and the student be very valuably convinced that $(1-1/n)^n$ is less than $\frac{1}{2}$ for all positive integral values of n, greater than $\frac{1}{3}$ for integral values greater than 5. Compare also Professor Bromwich's remarks in the *Gazette*, vol. iii., p. 87.]

The limiting values of $(1+1/n)^n$ and $(1-1/n)^{-n}$ are easily shown to be equal, for

$$(1-1/n)^{-n} = \frac{n}{n-1} \cdot \left(1 + \frac{1}{n-1}\right)^{n-1};$$

and their common value, an irrational number [cf. Chrystal, chap. xxviii., § 3], is denoted by e.

It is next necessary to establish in some way that the function $(1+1/x)^x$ tends to the value e when x tends to infinity *by*

* The student need not up to this stage have given any attention to the theory of infinite series; he will probably have had his attention drawn to the geometric infinite series and its application to recurring decimals.

continuous real variation. This is equivalent to the statement that the function $(1+x)^{1/x}$ is continuous through the value zero of the argument. Now, we know that this function has one and only one positive real value for each real value of x greater than -1, *the ideas of continuity being applied to justify this statement with reference to irrational values of* x [in the definition on the Dedekind-scheme of a power when the index is irrational]; also we know that it tends to the value e when x tends to zero by the reciprocals of the integers. The *assumption* that $(1+x)^{1/x}$ is continuous through the value zero of the argument can, therefore, hardly be regarded as violent.*

If then we write ξx for n in the function $(1+x/n)^n$ and suppose n to have positive integral values, ξ and x to have real values,

$$(1+x/n)^n=(1+1/\xi)^{\xi x}=\{(1+1/\xi)^{\xi}\}^x,$$

and therefore the limit to which it tends is e^x, *i.e.* $E(x)=e^x$.

Hence the function $E(z)$, defined by the convergent infinite series

$$1+z+\frac{z^2}{2!}+\ldots+\frac{z^r}{r!}+\ldots \textit{ ad inf.},$$

is called the exponential function, having the addition-theorem $E(x_1)\times E(x_2)=E(x_1+x_2)$ *for real values of* x_1 *and* x_2.

Derivative of e^x.

The inequalities

$$1-x<(1-x/n)^n<1-x+(1-1/n)\frac{x^2}{2}<1-x+\frac{x^2}{2},$$

which hold for positive real values of x if n has an integral value greater than x, give as limiting case the inequalities

$$1-x<e^{-x}<1-x+\frac{x^2}{2}, \textit{ i.e. } x>1-e^{-x}>x-\frac{x^2}{2}$$

for positive real values of x; and therefore

$$\operatorname*{Lt}_{h=+0} e^{x+h}\left(\frac{1-e^{-h}}{h}\right), \textit{ i.e. } \frac{d}{dx}.e^x, =e^x,$$

a very convincing and elementary proof of this fundamental fact.

Powers with Complex Indices.

It remains to exhibit how this theory gives a satisfactory means of defining a power for an index which is a complex number.

* If need is felt for further satisfaction, reference can be made to Chrystal, chap. xxv., § 13, where it is proved that the function $(1+1/x)^x$ increases steadily when an increasing system of positive *rational* values is given to x. Also to Harnack's *Introduction to the Calculus*, Bk. I., ch. vi., § 28, or to Gibson's *Calculus*, ch. v., § 48. *But some assumption similar in type to that suggested above must always be implicitly made*; cp. footnote p. 332, *Gazette*, July, 1906.

For real values of x_1 and x_2, we know that

$$(1+x_1/n)^n(1+x_2/n)^n \text{ and } \{1+(x_1+x_2)/n\}^n$$

each tend to the limit $e^{x_1+x_2}$ when n tends to infinity by positive integral variation. Hence

$$\{1+(x_1+x_2)/n+x_1x_2/n^2\}^n-\{1+(x_1+x_2)/n\}^n$$

tends to the limit zero for real values of x_1 and x_2.

Now
$$|\{1+(z_1+z_2)/n+z_1z_2/n^2\}^n-\{1+(z_1+z_2)/n\}^n|$$
$$=|\sum_{r=1}^{r=n} {}_nC_r.\{1+(z_1+z_2)/n\}^{n-r}.(z_1z_2/n^2)^r|$$
$$<\sum_{r=1}^{r=n} {}_nC_r\{1+(\rho_1+\rho_2)/n\}^{n-r}.(\rho_1\rho_2/n^2)^r,*$$

clearly, if ρ_1, ρ_2 denote $|z_1|$, $|z_2|$;

$$i.e. \quad <\{1+(\rho_1+\rho_2)/n+\rho_1\rho_2/n^2\}^n-\{1+(\rho_1+\rho_2)/n\}^n$$

and therefore tends to zero when n tends to infinity.

Hence the two functions

$$(1+z_1/n)^n(1+z_2/n)^n \text{ and } \{1+(z_1+z_2)/n\}^n$$

tend to the same limiting value; *i.e.* the function $E(z)$ has the addition-theorem

$$E(z_1)\times E(z_2)=E(z_1+z_2)$$

for all values of z_1, z_2 real or complex.

In particular $E(x+\iota y)=e^x.E(\iota y)$, for real values of x and y; and the complex function $E(\iota x)$, which is the limit of $(1+\iota x/n)^n$, may be suitably used to define the symbol $e^{\iota x}$ for real values of x.

The Trigonometric Series.

The real components of $E(\iota x)$ are given by the two convergent infinite series

$$C(x)\equiv 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots \textit{ ad inf.},$$

$$S(x)\equiv x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots \textit{ ad inf.}$$

And the fact that $\cos^r(x/n)$ and $\tan^r(x/n)/(x/n)^r$ tend to the value $+1$ when n tends to infinity, for all integral values of r from 1 to n, enables us to show that $e^{\iota x}=\cos x+\iota\sin x$: for

$$(\cos x/n+\iota\sin x/n)^n=\cos^n(x/n)\left\{1+\frac{\iota x}{n}\cdot\frac{\tan(x/n)}{x/n}\right\}^n;$$

* The device used here looks as if it might have a wide application; it is again applied t o theInfinite Products. See p. 362.

and, therefore, if we write $1+\epsilon_r$ for $\tan^r(x/n)/(x/n)^r$, so that ϵ_r is positive and tends to the limit zero when n tends to infinity,

$$\cos x+\iota\sin x=\cos^n(x/n)\left\{(1+\iota x/n)^n+\sum_{r=1}^{r=n}{}_nC_r\,.\,\epsilon_r\,.\left(\frac{\iota x}{n}\right)^r\right\};$$

and if η denotes the greatest of the ϵ's,

$$\left|\Sigma_nC_r\,.\,\epsilon_r\,.\left(\frac{\iota x}{n}\right)^r\right|<\eta\{(1+x/n)^n-1\}.$$

Now when n tends to infinity, $\cos^n(x/n)$ tends to the value $+1$, $(1+\iota x/n)^n$ tends to the value $e^{\iota x}$, $(1+x/n)^n$ to the value e^x and η to the value zero; therefore, $\cos x+\iota\sin x\equiv e^{\iota x}$ for real values of x.

Thus $\cos x=C(x)$, $\sin x=S(x)$ giving the infinite expansions for the sine and the cosine.

The Logarithmic Series.

Integration of the algebraic identity

$$(1+z)^{-1}\equiv 1-z+z^2-\ldots+(-)^{r-1}.z^{r-1}+(-)^r.z^r(1+z)^{-1}$$

leads to a proof of the fact that $\log_e(1+z)$ is the limit-function denoted by

$$z-z^2/2+\ldots+(-)^{r-1}.z^r/r+\ldots \textit{ ad inf.},$$

provided that $|z|<1$. [Cp. Gibson's *Introduction to the Calculus*, p. 165, Ex. 5.]

[The fact that this method provides so straightforward a rigorous demonstration of the logarithmic theorem is another link in the argument for taking the Calculus as the first branch of Higher Mathematics, making the Definite Integral our first application of the Limit-theorems given above.]

At this stage a short course on the general theory of Infinite Series would be introduced with advantage, before passing on to the other simple type of Infinite Sequence which appears naturally (in a manner analogous to that in which the Infinite Series have come) through

The Infinite Trigonometric Products.

Taking the generalised definitions of the sine- and cosine-functions to be expressed by the formulae

$$2\iota\sin z=E(\iota z)-E(-\iota z),\quad 2\cos z=E(\iota z)+E(-\iota z),$$

these functions have to do with limit-functions determined by the sequences

$$(1+\iota z/n)^n\mp(1-\iota z/n)^n.$$

Now it follows from the solution of $z^n=1$, that

$$(1+\iota z/n)^n-(1-\iota z/n)^n=2\iota z\prod_{r=1}^{r=n-1}\left\{1-\frac{z}{n\tan(r\pi/n)}\right\}$$
$$=2\iota z\prod_{r=1}^{r=\nu}\left\{1-\frac{z^2}{n^2\tan^2(r\pi/n)}\right\},$$

where ν denotes $(n-1)/2$ or $(n-2)/2$, according as n is odd or even.

The fact that, throughout the product, $r<n/2$ provides that $n\tan(r\pi/n)>r\pi$; and, *for finite values of* r, $n\tan(r\pi/n)$ tends to $r\pi$ as a limit when n tends to infinity: thus, we may write

$$\prod_1^r\left\{1-\frac{z^2}{n^2\tan^2(r\pi/n)}\right\}=\{P(z,k)+\eta(z,k,n)\}.R(z,k,n),$$

where

$$P(z,k)\equiv\prod_1^k(1-z^2/r^2\pi^2),\quad R(z,k,n)\equiv\prod_{k+1}^{\nu}\left\{1-\frac{z^2}{n^2\tan^2(r\pi/n)}\right\},$$

and $\eta(z,k,n)$ tends to zero when n tends to infinity, *for finite values of* k *and* z.

But if the expansion of $R(z,k,n)$ be denoted by

$$1-a_1.z^2+a_2.z^4-\dots,$$

the a*'s denoting positive real constants,*

$$|1-R(z,k,n)|<a_1.\rho^2+a_2.\rho^4+a_3.\rho^6+\dots$$
$$<\prod_{k+1}^{\nu}\left\{1+\frac{\rho^2}{n^2\tan^2(r\pi/n)}\right\}-1^*$$
$$<\prod_{k+1}^{\nu}(1+\rho^2/r^2\pi^2)-1,$$

which, provided that $k\pi>\rho$,

$$<\{1/\Pi(1-\rho^2/r^2\pi^2)\}-1,$$

and therefore

$$<\pi^2/(\pi^2-\rho^2.\sum_{k+1}^{\nu}1/r^2)-1.$$

[See the inequalities, p. 357.]

Thus it follows, from the algebra of the convergent series $\sum^{\infty}1/r^2$, that

$$R(z,k,n)=1+\gamma(z,k,n),$$

where $\gamma(z,k,n)$ is a function whose value is diminished indefinitely when a system of indefinitely increasing values is given to k, z having any finite value and n any integral value so great that $n>2k$.

Also, an obvious slight variation of the above analysis proves that the "infinite product" $\prod_1^{\infty}(1-z^2/r^2\pi^2)$ is "convergent," *i.e.*

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* Cp. p. 360 of this number.

that the sequence, $\prod_1^n(1-z^2/r^2\pi^2)$, involved determines a limit-function, which may be denoted by $P(z)$; hence

$$P(z, k)=P(z)+\delta(z, k),$$

where $\delta(z, k)$ is a function which, for any finite value of z, tends to zero when k tends to infinity.

Thus, finally,

$$\prod_1^\nu\left\{1-\frac{z^2}{n^2\tan^2(r\pi/n)}\right\}=\{P(z)+\delta(z, k)+\eta(z, k, n)\}\{1+\gamma(z, k, n)\};$$

and it follows (cp. p. 359) that, corresponding to any assigned finite positive real number, ϵ, *however small*, a value (N) of n can be chosen (greater than a value of k chosen first), such that

$$\left|\prod_1^\nu\left\{1-\frac{z^2}{n^2\tan^2(r\pi/n)}\right\}-P(z)\right|<\epsilon, \text{ provided } n>N.$$

Hence

$$\begin{aligned}\sin z &= z\,.\,P(z)\\ &= z\,.\prod_1^\infty(1-z^2/r^2\pi^2).\end{aligned}$$

Similarly, from the algebraic identity,

$$(1+\iota z/n)^n+(1-\iota z/n)^n\equiv 2\prod_1^\nu\left\{1-\frac{z^2}{n^2\tan^2(2r-1\,.\,\pi/n)}\right\},$$

where ν denotes $(n-1)/2$ or $n/2$ according as n is odd or even, it follows by an exactly similar process that

$$\cos z=\prod_1^\infty\{1-4z^2/(2r-1)^2\pi^2\}.$$

And it is clear that the method might be applied to the fractional infinite series for cosec z, sec z, cot z, and tan z.*

The investigation given here seems to provide a serious criticism of Mr. Hardy's dogma, that the use of the function $(1+z/n)^n$ in the Exponential theory "is instructive and important, but essentially accessory and not fundamental, and to base the exponential theorem on it is *logically quite wrong*" [italics supplied]; and that the factor-theorem "is *really* difficult," providing "a *crux* in the teaching of trigonometry" [*Gazette*, p. 285].

That the line of argument is logical will hardly be denied, and that it gives valuable insight into the infinite sequence as an instrument will probably be admitted by anyone who cares to devote the best part of a term's teaching to it. The amount

* For another method of establishing these results, see a paper by the writer of this article in the *Proc. Edin. Math. Soc.*, vol. xxii.; the treatment there is of the real case, but the analysis is suggestive in the present connection and instructive to the beginner.

of detail is perhaps a little alarming, but there is only one dominating idea, which is not hard, for the whole theory. And the writer repeats that a knowledge of this theory is not for the immature schoolboy mind; very serious damage has been done to mathematical training by attempts to gloss over the inherent difficulties which lie at the foundations of the Higher Trigonometry.

D. K. PICKEN.

ON A PROBLEM IN MECHANICS AND THE NUMBER OF ITS SOLUTIONS.

To find the position of equilibrium of a particle which is acted on by forces of given magnitudes directed towards given points.

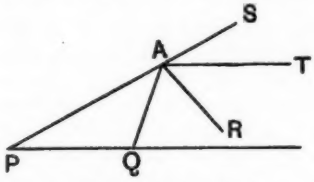
This problem was suggested by Mr. P. A. Hillhouse, and arose in the first instance from the consideration of forces acting in a crane. The following note does not offer a solution of the problem, but contains a proof by Mr. A. L. Jones that there is at most one position of equilibrium when the forces are all of one sign, *i.e.* either all tensions or all thrusts. This is followed by an investigation of the total number of solutions, real and imaginary, for all combinations of thrusts and tensions, when the magnitude only of each force, and not its sign, is given. This number is surprisingly large. It would of course be of more practical interest to find the number of *real* solutions for each particular combination of thrusts and tensions, but I have only succeeded in doing this when the given points are in a straight line, and in one or two other special cases. The results for these are stated without proof at the end of the note.

When the forces are all of one sign the single real position of equilibrium could be approximately found experimentally by means of threads, all knotted together at one end, which are passed through small smooth rings at the given points, and carry at their free ends weights proportional to the given forces. The position taken by the knot will then be the position of equilibrium.

Mr. Jones' proof (for forces all of one sign) depends on the simple lemma that *two equal positive forces in the directions PA, AQ have a resultant whose direction makes an acute angle with the direction of PQ.* For the resultant AR bisects the angle QAS, formed by AQ and PA produced; and the line through A in the direction PQ makes with AR an angle less than half QAS, that is, an acute angle. The resultant cannot vanish unless A is on the line PQ and external to PQ; and the lemma holds if A is internal to PQ.

Suppose that P is a point such that given positive forces

$t_1, t_2, \dots, t_n$ acting from P towards given points $A_1, A_2, \dots, A_n$ are in equilibrium, and that if possible a second point Q exists for which the same is true. Then the system of forces $t_1, t_2 \dots, t_n$ along $PA_1, PA_2, \dots, PA_n$, and the system of forces $t_1, t_2, \dots, t_n$ along $A_1Q, A_2Q, \dots, A_nQ$, are in equilibrium separately, and therefore in equilibrium jointly. But t_r along PA_r and t_r along A_rQ have a resultant which makes an acute angle with PQ; and the two systems are therefore equivalent to a single system of positive forces all of which make acute angles with PQ. This system cannot be in equilibrium unless all the forces vanish; and this cannot happen unless $A_1, A_2, \dots, A_n$ are collinear, in which case there is either no position of equilibrium or an infinite number of such positions. If the given points are not collinear there can only be one real position of equilibrium at most.



If forces $t_2, t_3, \dots, t_n$ along $A_1A_2, A_1A_3, \dots, A_1A_n$ have a resultant t_1, then A_1 is the position of equilibrium. The necessary and sufficient conditions that there may be a position of equilibrium other than $A_1, A_2, \dots, A_n$ are that the resultant of the forces $t_1, \dots, t_{r-1}, t_{r+1}, \dots t_n$ along $A_rA_1, \dots A_rA_{r-1}, A_rA_{r+1}, \dots A_rA_n$ should be greater than t_r, for all values of r from 1 to n. For if these n conditions hold, and the particle be placed at any one of the points $A_1, A_2, \dots, A_n$, it will be drawn away by the forces which act upon it; and the forces cannot send the particle away to infinity, if they are tensions. For tensions the position of equilibrium is stable, and for thrusts unstable.

A slight extension of the above proof shows that if at one of the given points, say at A_1, the resultant of the forces $t_2, t_3, \dots, t_n$ along $A_1A_2, A_1A_3, \dots, A_1A_n$ is equal to or less than t_1, then for any other point A_r the resultant of a like set of forces is greater than t_r. From this it follows that $n-1$ of the n conditions mentioned above hold of necessity.

We will now consider the problem analytically and find the total number of solutions, real and imaginary, when the absolute magnitudes only of the forces are given. We consider first the case in which the given points are in a plane. Let a_r, b_r be the rectangular coordinates of A_r $(r=1, 2, \dots n)$, and x, y the coordinates of the particle P, which is in equilibrium under the system of forces $t_1, t_2, \dots t_n$ along $PA_1, PA_2 \dots PA_n$. The equations for equilibrium, by resolving along the axes, are

$$\frac{t_1(x-a_1)}{r_1}+\frac{t_2(x-a_2)}{r_2}+\ldots+\frac{t_n(x-a_n)}{r_n}=0,$$

$$\frac{t_1(y-b_1)}{r_1}+\frac{t_2(y-b_2)}{r_2}+\ldots+\frac{t_n(y-b_n)}{r_n}=0,$$

where $$r_p^2=(x-a_p)^2+(y-b_p)^2.$$

Assuming $r_1, r_2, \ldots r_n$ to be always positive,* we cannot in general find the solutions of these equations without at the same time finding the solutions for all the 2^{n-1} combinations of thrusts and tensions among the given forces. We shall, however, assume that $r_1, r_2, \ldots r_n$ are capable of either sign, while $t_1, t_2, \ldots t_n$ remain invariably positive, as this gives us the same system of equations, and leaves the values of the unknowns x, y, r_1, $r_2, \ldots r_n$ unrestricted. In order to discuss the infinite (as well as the finite) solutions we introduce another unknown u so as to make the equations homogeneous. Then the system of equations for the ratios of u, x, y, $r_1, \ldots r_n$ is

$$\frac{t_1(x-a_1u)}{r_1}+\ldots+\frac{t_n(x-a_nu)}{r_n}=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{t_1(y-b_1u)}{r_1}+\ldots+\frac{t_n(y-b_nu)}{r_n}=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$(x-a_1u)^2+(y-b_1u)^2=r_1^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(x-a_nu)^2+(y-b_nu)^2=r_n^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots(n+2)$$

Equations (1), (2) are written in a fractional form so as to be less cumbersome, but can be at once made integral by multiplying by $r_1r_2\ldots r_n$. Thus there are two equations of degree n, and n equations of degree 2, and the total number of solutions, taking account of infinite and multiple solutions, is n^22^n. There are, however, a large number of irrelevant solutions which do not satisfy the conditions of the problem; their number must be found and subtracted from n^22^n in order to find the true number of solutions of the problem.

In particular, all infinite solutions (those in which $u=0$) are irrelevant, and must be rejected; it will be found that there are only two different infinite solutions, but that each is multiple to a high degree. In addition some multiple and non-multiple finite solutions are also irrelevant. In general the relevant solutions are all non-multiple and finite.

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*The function $t_1r_1+t_2r_2+\ldots+t_nr_n$ is the potential of the system of forces, and has a stationary value at a position of equilibrium P. The curve of the family

$$t_1r_1+t_2r_2+\ldots+t_nr_n=\text{constant},$$

which passes through P, has a node at P; and a node is a position of equilibrium. If two positions of equilibrium move up to coincidence the node becomes a cusp.

The chief difficulty is to find the multiplicity of the multiple solutions. To do this we make use of a principle which is seldom applied: *In a given system of n non-homogeneous equations for n unknowns* $x_1, \dots x_n$ *the multiplicity of the solution*

$$x_1 = x_2 = \dots = x_n = 0$$

is the product of the degrees of the terms of lowest degree in the equations, provided that these lowest terms when equated to zero have no proper solution, that is, no solution other than $x_1 = x_2 = \dots = x_n = 0$. Again the multiplicity of the solution $x_1 = a_1$, $x_2 = a_2, \dots x_n = a_n$ is found by the same rule after substituting $x_1 + a_1, \dots x_n + a_n$ for $x_1, \dots x_n$ in all the equations. Finally the rule is applied to a system of *homogeneous* equations by making them non-homogeneous, by putting a non-vanishing unknown, of which there must be at least one in any proper solution, equal to unity.

The rule can only be applied in what is called the "simple case," viz. when the equations can be so written that the lowest terms when equated to zero have no proper solution. For example, the multiplicity of the solution $x_1 = x_2 = 0$ in the equations $x_1 - x_2 + x_1^2 = 0$, $x_1^2 - x_2^2 + 2x_1^2 x_2 = 0$ is 4; for, on multiplying the first by $x_1 + x_2 - x_1^2$, and subtracting from the second, we get $x_1^4 = 0$, which may be taken in place of the second equation. The equations being written in any definite order, each equation may be modified as above by means of previous equations, but not by means of succeeding equations, since such modification does not affect the number or the multiplicity of the finite solutions. In the case of the equations $x_1^2 - x_2^2 + x_1^3 = 0$, $x_1^3 - x_2^3 + x_2^4 = 0$, the proviso cannot be fulfilled and the rule cannot be applied. This example belongs to what is called the "general case"; but all the solutions of the system of equations (1) to $(n+2)$ come under the "simple case."

We can now proceed to find all the irrelevant solutions of the equations (1) to $(n+2)$. Consider first the infinite solutions. On putting $u = 0$, the equations become $x\Sigma\frac{t}{r} = y\Sigma\frac{t}{r} = 0$, $x^2 + y^2 = r_1^2 = \dots = r_n^2$. Hence either $x = y = 0$, which gives no proper solution; or else $\Sigma\frac{t}{r} = 0$, which requires all the r's to vanish, unless one of the expressions $t_1 \pm t_2 \pm \dots \pm t_n$ vanishes, which we shall assume not to be the case. Thus there are only two different infinite solutions, viz. $u = r_1 = r_2 = \dots = r_n = 0$, $x = \pm \iota y$.

To find the multiplicity of the infinite solution corresponding to $x + \iota y = 0$, put $x + \iota y = v$ and $x - \iota y = 1$, thus making the system non-homogeneous with $v, u, r_1, \dots r_n$ as unknowns, and

$$u = v = r_1 = \dots = r_n = 0$$

as the solution whose multiplicity is required. Now from equations (3), (4) we can express u and v as quadratic functions of u, v, r_1, r_2, and by substituting them in equations (5) to $(n+2)$, the lowest terms in these become of degree 2. Also (1), (2) become $(v+1)\Sigma\frac{t}{r}=2u\Sigma\frac{at}{r}$, $(v-1)\Sigma\frac{t}{r}=2\iota u\Sigma\frac{bt}{r}$, of which the second may be replaced by $v\Sigma\frac{t}{r}=u\Sigma\frac{(a+\iota b)t}{r}$, which is of degree $n+1$ when made integral, and when for u, v are substituted the quadratic functions to which they are equal. The product of the degrees of the lowest terms in the equations is now

$$(n-1)(n+1)2^{n-2}.$$

This is the multiplicity of each of the infinite solutions. The number of finite solutions is therefore

$$n^2 2^n-(n^2-1)2^{n-1}=(n^2+1)2^{n-1}.$$

As regards these we may put $u=1$, since $u \neq 0$.

The irrelevant finite solutions are of two kinds. There are 2^{n-1} solutions for which $x=a_p$, $y=b_p$, $r_p=0$; for these values satisfy equations (1), (2), $(p+2)$, and the other equations have 2^{n-1} solutions. Each of these solutions is of multiplicity 2, owing to equation $(p+2)$. This accounts for $n2^n$ irrelevant solutions. Again there are 2^{n-1} finite solutions for which $r_p=r_q=0$; for these values satisfy (1), (2), and leave all the other equations quadratic except one of the equations $(p+2)$, $(q+2)$; which give a linear equation by subtraction.* This accounts for $\frac{1}{2}n(n-1)2^{n-1}$ irrelevant solutions. Assuming that this exhausts the number of irrelevant solutions we find the number of relevant solutions to be

$$(n^2+1)2^{n-1}-n2^n-n(n-1)2^{n-2}=(n-1)(n-2)2^{n-2}.$$

The number of solutions for x, y is only half of this, since to each solution $x, y, r_1, r_2, \ldots r_n$ there corresponds another x, y, $-r_1, -r_2, \ldots -r_n$ in which x, y, have the same values. The final result is that *the total number of solutions of the problem for n co-planar forces is* ${}_{n-1}C_2 2^{n-2}$. I have verified by other methods that this number is correct when $n=3$, 4, or 5, which goes to show that the method of exhaustion followed above is complete. The number is the total number of solutions, real and imaginary, for all the 2^{n-1} combinations of thrusts and tensions among the given forces. The average number of solutions for each combination is therefore $\frac{1}{4}(n-1)(n-2)$.

When the given points are co-planar there is a case of exception in which the number of solutions is infinite. This occurs when

*This diminution to half the normal number of solutions is accounted for by the fact that the other half are at infinity. The infinite solutions have been previously considered separately.

the points all lie on a circle, and a point on the circle, other than one of the given points, is a position of equilibrium. In that case every point on the circle is a position of equilibrium. Also (when the points are on a circle) if there is any real position P of equilibrium outside the plane, then every point on a certain circle through P is a position of equilibrium. This circle and the given circle lie in perpendicular planes, and each cuts the plane of the other in a pair of inverse points.

A similar method can be applied to find the total number of solutions of the problem for n given points in ordinary space. The process is longer and more difficult, and we shall give only a short sketch of it.

In this case we have $n+3$ equations for the ratios of the $n+4$ unknowns $u, x, y, z, r_1, r_2, \ldots r_n$, viz.

$$\frac{t_1(x-a_1u)}{r_1}+\ldots+\frac{t_n(x-a_nu)}{r_n}=0, \quad \ldots\ldots(1)$$

$$\frac{t_1(y-b_1u)}{r_1}+\ldots+\frac{t_n(y-b_nu)}{r_n}=0, \quad \ldots\ldots(2)$$

$$\frac{t_1(z-c_1u)}{r_1}+\ldots+\frac{t_n(z-c_nu)}{r_n}=0, \quad \ldots\ldots(3)$$

$$(x-a_1u)^2+(y-b_1u)^2+(z-c_1u)^2=r_1^2, \quad \ldots\ldots(4)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(x-a_nu)^2+(y-b_nu)^2+(z-c_nu)^2=r_n^2. \quad \ldots\ldots(n+3)$$

These equations have a singly infinite system of solutions, together with a finite number of other solutions which include all the relevant solutions. In geometrical language equations (2) to $(n+3)$ have as their complete intersection a composite curve of degree $n^2 2^n$, part of which lies in (1), and the remaining part, which we shall call the *remainder curve*, cuts (1) in a finite number of points, some giving relevant and some irrelevant solutions. We must first find the degree of the remainder curve.

Part of the intersections of (2) to $(n+3)$ consists of an *infinity curve* of degree 2, which lies in (1), viz. the curve

$$u=r_1=r_2=\ldots=r_n=0, \quad x^2+y^2+z^2=0.$$

For, on putting $u=0$, we have

$$y\Sigma\frac{t}{r}=z\Sigma\frac{t}{r}=0, \quad x^2+y^2+z^2=r_1^2=r_2^2=\ldots=r_n^2.$$

Hence either $y=z=0$, giving 2^n points, none of which lie in (1); or $\Sigma\frac{t}{r}=0$, which requires all the r's to vanish, assuming that no expression of the form $t_1 \pm t_2 \pm \ldots \pm t_n$ vanishes. This gives the infinity curve. To find its multiplicity in the intersection of (2) to $(n+3)$ we put $z=1$, and change x, y to $x+x_1$,

$y+y_1$, where $x_1^2+y_1^2+1=0$. From equations (4), (5) we can then express u and xx_1+yy_1 as quadratic functions of u, x, y, r_1, r_2; and by substituting them in (6) to $(n+3)$ the lowest terms in these last become of degree 2. Also equations (2) (3) become $(y+y_1)\Sigma\frac{t}{r}=u\Sigma\frac{bt}{r}$, $\Sigma\frac{t}{r}=u\Sigma\frac{ct}{r}$, the former of which may be changed to $u\Sigma\frac{bt}{r}=(y+y_1)u\Sigma\frac{ct}{r}$, whose lowest terms are of degree $n+1$, when we substitute for u the quadratic function to which it is equal. The infinity curve is therefore of multiplicity $(n+1)(n-1)2^{n-2}$, and counts as of degree $(n^2-1)2^{n-1}$, in the intersection of (2) to $(n+3)$. The complete *finite* intersection of (2) to $(n+3)$ is therefore a curve of degree

$$n^2 2^n-(n^2-1)2^{n-1}=(n^2+1)2^{n-1}.$$

This finite curve includes $\frac{1}{2}n(n-1)$ curves which lie in (1) viz. those for which any two of the r's vanish. On putting $u=1$ and $r_1=r_2=0$, equations (4), (5) are equivalent to one quadratic and one linear equation, while (6) to $(n+3)$ are quadratic. Hence each of the $\frac{1}{2}n(n-1)$ curves is of degree 2^{n-1}, and the *remainder curve* is of degree

$$(n^2+1)2^{n-1}-n(n-1)2^{n-2}=(n^2+n+2)2^{n-2},$$

which cuts (1) in $n(n^2+n+2)2^{n-2}$ points altogether.

All the points in which the remainder curve cuts (1) on the infinity curve and the $\frac{1}{2}n(n-1)$ curves give irrelevant solutions, and also a few other points. The remainder curve meets infinity $(u=0)$ in $(n^2+n+2)2^{n-2}$ points, of which only 2^n (corresponding to $y=z=0$) are not on the infinity curve. The other points lie on the infinity curve; but at each of these points the remainder curve touches $u=0$, so that it has only $(n^2+n-2)2^{n-3}$ points of intersection with the infinity curve. At each of the $(n^2+n-2)2^{n-3}$ points the remainder curve meets (1) in a point of multiplicity $n+1$. This accounts for $(n+1)(n^2+n-2)2^{n-3}$ irrelevant infinite solutions.

At any finite point at which three of the r's vanish, the complete intersection of (2) to $(n+3)$ has four branches, three of which belong to curves for which two of the r's vanish, and the fourth to the remainder curve. There are 2^{n-2} such finite points at which any one set of three r's vanish, and at each the remainder curve meets (1) in a point of multiplicity 2. This accounts for $\frac{1}{6}n(n-1)(n-2)2^{n-1}$ irrelevant finite solutions.

The remainder curve also cuts the curve corresponding to $r_p=r_q=0$ in 2^{n-1} finite points at which no other r vanishes, viz. the points for which $\frac{y-b_p}{z-c_p}=\frac{y-b_q}{z-c_q}=\frac{b_p-b_q}{c_p-c_q}$. This accounts for $\frac{1}{2}n(n-1)2^{n-1}$ irrelevant solutions.

Also the remainder curve meets (1) in 2^{n-1} points corresponding to $x=a_p$, $y=b_p$, $z=c_p$, $r_p=0$, each of which is of multiplicity 2. This accounts for $n2^n$ irrelevant solutions.

Hence we find the number of relevant solutions to be

$$\begin{aligned} &n(n^2+n+2)2^{n-2}-(n+1)(n^2+n-2)2^{n-3} \\ &\qquad -\tfrac{1}{6}n(n-1)(n-2)2^{n-1}-n(n-1)2^{n-2}-n2^n \\ &=\tfrac{1}{6}(n+3)(n-1)(n-2)2^{n-2} \\ &=\tfrac{1}{6}(n-1)(n-2)(n-3)2^{n-2}+(n-1)(n-2)2^{n-2}. \end{aligned}$$

The number of solutions for x, y, z is half of this.

Hence the total number of solutions of the problem for n forces in ordinary space is ${}_{n-1}C_2 2^{n-2}+{}_{n-1}C_3 2^{n-3}$.

This result is correct also if the given points are all in one plane, for then the total number of solutions in the plane is ${}_{n-1}C_2 2^{n-2}$, and out of the plane ${}_{n-1}C_3 2^{n-3}$. This enables us to guess the answer to the problem for n forces in space of m dimensions. We may in fact assume the answer to be

$${}_{n-1}C_2 2^{n-2}+{}_{n-1}C_3 2^{n-3}+\ldots+{}_{n-1}C_m 2^{n-m}.$$

For $n+1$ forces in space of n dimensions the number of solutions is $2(3^n-\overline{n+2}\,2^{n-1})$, a result which can be verified in other ways.

As regards the number of real solutions, when the given points are not collinear, I can only mention two results: (i) that for four points in ordinary space the fourteen solutions can all be real; (ii) that for $n+1$ points in space of n dimensions, when the distance between every pair of points is the same, and the forces are all equal, there is one and only one real solution for each combination of thrusts and tensions. If, however, $n+1$ is even, the solutions corresponding to an equal number of thrusts and tensions are at infinity and of multiplicity 3.

In the case of a tetrahedron of which the base is equilateral, and the three faces have the same inclination 2α to the base, if the force to (or from) the vertex is t_0, and to (or from) each of the base angles is t, there will be fourteen real solutions provided t_0/t lies between 1 and $3\sin\alpha(1-3\sin^2\alpha\cos^2\alpha)^{\frac{1}{2}}$. (Thus for a rectangular tetrahedron t_0/t must lie between ·975... and 1). When all four forces are tensions, or when three are tensions and one a thrust, there is one solution in each case, and when two are tensions and two are thrusts there are three solutions in each case.

When the given points are collinear the number of real positions of equilibrium in any plane through the line of collinearity can be stated completely, provided no expression of the form $t_1\pm t_2\pm\ldots\pm t_n$ vanishes. Let $A_1, A_2, \ldots A_n$ be the given points in order, and $t_1, t_2, \ldots t_n$ the forces (given in magnitude and sign) directed towards them. As the sign of one force may be chosen either positive or negative, we *assume that* t_1 *is positive.* Then the number of real positions of equilibrium on each side of the

line is the excess of the number of positive maxima over the number of positive minima in the series of quantities

$$(-S_n),\ S_1,\ S_2,\ \dots S_n,\ (-S_1),$$

where

$$S_1 = -t_1 + t_2 + \dots + t_n = \Sigma t - 2t_1,$$
$$S_2 = -t_1 - t_2 + \dots + t_n = S_1 - 2t_2,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$S_n = -t_1 - t_2 - \dots - t_n = S_{n-1} - 2t_n.$$

By a maximum (or minimum) quantity is meant one which is greater (or less) than each of the adjacent quantities. The rule is applied to $S_1, S_2, \dots S_n$ only, S_1 and S_n being counted as minima if less than the adjacent quantity, but not as maxima if greater than the adjacent quantity, since $S_1 < -S_n$, $S_n < -S_1$.

Thus if there are $2n+1$ equal forces alternately pushing and pulling, there are n real positions of equilibrium on each side of the line.

If $t_1, t_2, \dots t_n$ are the absolute magnitude of the forces, and positive, the total number of real positions of equilibrium for all possible combinations of thrust and tension is the excess of the total number of positive signs over the total number of negative signs in all the expressions $-t_1 \pm t_2 \pm \dots \pm t_n$ which have positive values. This result (like the last) is independent of the distances between the given points, and (unlike the last) remains unchanged if the order of the points is changed.

The greatest possible number of real positions of equilibrium is found by subtracting each coefficient in the expansion of $(1+x)^{n-1}$ from the greatest coefficient, and adding all the results. This is counting the solutions on both sides of the line; and may be compared with the total number of solutions, real and imaginary, viz. $(n-1)(n-2)2^{n-3}$. The ratio is roughly that of 3 to $n^{\frac{3}{2}}$.

F. S. MACAULAY.

REVIEWS.

The Theory of Sets of Points. By W. H. YOUNG, M.A., Sc.D., and GRACE CHISHOLM YOUNG, Phil. Doc. (Gött.). (Cambridge: at the University Press. 1906. Pp. xii + 316.)

The theory of aggregates may roughly be divided into two parts: the one, which arose out of the profound study of integration, is associated with the names of Riemann, Hankel, Smith, Harnack, P. du Bois-Reymond, Lebesgue, and Young; while the other, which arose from the need for a secure basis of the theory of analytic functions, has, as characteristics, Weierstrass' and Cantor's theories of irrationals, Cantor's theory of derivatives of point aggregates, Cantor's theory of the 'powers' (*Mächtigkeiten*) of aggregates, Cantor's theory of transfinite numbers, and the modern developments of this last theory. The two parts are by no means disconnected; thus, Cantor has greatly

contributed to the theory of the "content" of an aggregate (the essential notion in the conception of integration), and Dr. Young, whose past and present work has been mainly concerned with what may be called the Riemann-direction, has given some excellent chapters on Cantor's ideas and numbers.

To quote the author's preface: "In subjects as wide apart as Projective Geometry, Theory of Functions of a Complex Variable, the Expansions of Astronomy, Calculus of Variations, Differential Equations, mistakes have in fact been made by mathematicians of standing, which even a slender grasp of the Theory of Sets of Points would have enabled them to avoid." This should raise interest in the theory even among mathematicians who do not concern themselves with fundamental questions, while, for those who do, this theory has, of course, shown itself to be indispensable.

On account of both the novelty of the subject as material for a text-book and the independence and originality of its treatment, the work of Dr. and Mrs. Young deserves the highest praise; in this notice, I have thought it would be more useful, instead of praising the many excellences of the book in detail, to direct attention to those few parts which, in my opinion, require modification or discussion.*

The correspondence between the real numbers and the points of a straight line is dealt with on pp. 9-15. By a fairly well-known extension of von Staudt's method of forming a projective number-scale, combined with the Cantor-Dedekind axiom, the result is reached (p. 14) that the continuum of real numbers and the linear continuum can be brought into a (1, 1)—correspondence, maintaining the order. Two points should be remarked. Firstly, Cantor stated his axiom (the date on p. 14 should be 1872 instead of 1878) in a form applying to the *metrical* geometry of the straight line. Secondly, all we know of that projective scale is that it is of type θ,†—and series of *some* (which are not *all*) real numbers can be of this type (Harnack, Bettazzi‡). Thus it seems that purely ordinal characteristics do not describe our notion of the straight line, but that a line (as a substratum to a point-set) is a set of points in *metrical* (not merely ordinal) correspondence with the system of real numbers. The difference between sets of points on a substratum and a set of intrinsically determined order is emphasised on pages 22-23, 127, and 287.

In the definition (p. 16)§ of what is meant by a 'set of points,' we read: "a linear set of points consists of points of a straight line determined by a certain law which is such that (1) every point of the straight line either belongs to the set or does not, but not both, nor neither; . . . ." This condition seems so self-evident to the student that its meaning cannot be grasped at this stage. The importance of the condition lies in the fact that we can frame certain 'definitions' which *appear* to define valid classes ('sets'), and yet such a 'set' both

* There is (pp. 1-5) a short account of the system of real numbers as defined by (Cantor's) 'sequences.' I hope to give, in the near future, a brief discussion of what one can mean by the phrase 'definition of a number by a sequence' in this *Gazette* (cf. Russell, *The Principles of Mathematics*, vol. i., 1903, pp. 270-286).

† Cf. Russell, *op. cit.*, pp. 353, 384 *sqq.* ‡ *Ann. di mat.*, t. XVI. (1888-9), pp. 49-60.

§ An analogous definition is on p. 170.

is and is not a member of itself. Such 'sets' are Russell's 'class of the x's such that x is not a member of itself' and the class known as the 'cardinal number of all the ordinal numbers.'*

It is logically better to separate the idea of 'potency' (power, or cardinal number) from that of measurement (cf. pp. 33, 76), for, as Russell† has shown, the latter idea is irrelevant to this part of mathematics, at least.

In the description (pp. 124-127) of Bernstein's diagrammatic representation of simply-ordered enumerable series, reference should, perhaps, be made to the prototype of this,—Schröder's representation by a 'matrix' of a binary relative.‡

There seems to be some confusion in the note on p. 159. The number ω^ω is not the number ω exponentiated by ω in any sense analogous to exponentiation as defined for cardinal numbers. The definitions of ordinal exponentiation given by Hausdorff and myself,§ on the contrary, are.

In Ex. 5 on p. 152, the symbolism $\operatorname{Lt}_{n=\infty} n!$ is wrong; for the meaning of this is: The least cardinal which is greater than all the (finite) cardinals $n!$; and this is a; whereas what we want is the product of *all* the finite cardinals, and this is equal to c.

In the discussions of the law of arbitrary choice and the multiplicative axiom (pp. 147, 289-291), it appears to me that Bernstein's conception of 'multiple equivalence' is irrelevant. Given any not-null class u, we can state certain propositions beginning 'x is a u,' and it is irrelevant in how many ways x can be chosen: the multiplicative axiom is necessary in order to be able to say that '$x, y, z, \ldots$ (an infinity) are different members of u' has any meaning at all (given, of course, that u is infinite). A case of this axiom, it should be stated, is involved in the definition (p. 151) of the product of an infinity of cardinals.

There are some small misprints: p. 19 (note), for 1881 read 1882, and for 1892 read 1882; pp. 124, 134, 301, 'Huntington' is spelt wrongly; on p. 155, for 2ω read $\omega . 2$, and so on; on page 191 (note) the reference to Bolzano should be "Rein analytischer Beweis...," 1817 (lately republished, with notes, in Ostwald's *Klassiker der exakten Wissenschaften*); on p. 295, 'Lebesgue' is spelt wrongly, and on p. 299, 'Beiträge'; and (p. 155) a number is not a 'symbol.'

In the admirable Bibliography (pp. 295-309), it should, I think, be remarked that the French translations of Cantor in vol. 2 of *Acta Math.* are not always full; that the work mentioned in the 10th reference to Cantor is included in that mentioned in the 9th; and references to Dedekind's "Stetigkeit und irrationale Zahlen" of 1872, G. H. Hardy's important paper on absolutely convergent series in *Proc. Lond. Math. Soc.*, for 1905, Pringsheim's article on irrational numbers in the *Encykl. der Math. Wiss.*, and Russell's papers in *Mind of* 1905 (on "Denoting") and *Proc. Lond. Math. Soc.*, for 1906, should be given.

PHILIP E. B. JOURDAIN.

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* Cf. pp. 145-147, especially the suggestive ll. 16-25 on p. 146. † *Op. cit.*, pp. 111-116.
‡ *Algebra und Logik der Relative*, Leipzig, 1895, pp. 42-68.
§ *Mess. of Math.*, May, 1906, pp. 13-16.

Tableaux logarithmiques. A et B. By DR. A. GUILLEMIN. Pp. 32 of explanatory matter, with two pockets containing the two tables. Paris, Félix Alcan, 1906. Price, 4 francs complete. 2 francs for separate parts.

Let N be a number whose logarithm is required, and let m be a near number whose logarithm is given in a table. Then if $N = m(1+\alpha)$ the quantity to be added to the known $\log m$ in order to obtain the required $\log N$ is $\log(1+\alpha)$, and is a quantity of the same order of magnitude as α, when α is small.

This property is utilised by Dr. Guillemin to enable logarithms and antilogarithms to be obtained to 6 and 9 decimals respectively by means of his tables A and B, each table containing only 1,000 entries, and the larger table measuring only roughly 1 ft. 6 in. × 1 ft. 3 in. The tables are really tables of antilogarithms rather than of logarithms. The logarithms in the second column (L) are the successive decimal numbers of three figures from 000 to 999; the corresponding antilogarithms in the first column are given to six places in Table A and to nine places in Table B. The third columns give the values of $\log \alpha$ to 3 and 6 places respectively, for which the corresponding values of $\log(1+\alpha)$ are entered in the second columns.

Suppose then I want to find $\log 3{\cdot}141593$ from Table A to six places. The nearest number in the table is 3140509, of which the logarithm is given as ·497, and these are the first three figures of the required logarithm. The process stands thus:

$$N = 3{\cdot}141593 \qquad \log N = \log m + \log(1+\alpha)$$
$$m = 3{\cdot}140509 \qquad \log m = {\cdot}497$$

By subtraction

$$N - m = m\alpha = 0{\cdot}001084.$$

I then look for $\log 0{\cdot}001084$ in the tables and find $\log 1083927 = {\cdot}035$, which is the value required for $\log m\alpha$ *to three decimal places.* The work now stands thus:

$$\log m\alpha = \bar{3}{\cdot}035$$
$$\log m = {\cdot}497 \text{ (above)},$$
$$\therefore \log \alpha = \bar{4}{\cdot}538.$$

I then look for $\bar{4}{\cdot}538$ in the column headed $\log \alpha$ and find it opposite 150 in the column L. This means that the corresponding value of $\log(1+\alpha)$ is ·000150, and adding this to the value of $\log m$ the final result is $\log 3{\cdot}141{,}593 = {\cdot}497150$.

In the second table (Table B) the antilogarithms are given to 9 decimal places and the values of $\log \alpha$ to 6 decimal places. To obtain logarithms correct to 9 decimal places it might be thought at a first glance that it would only be necessary to repeat twice the operations required with six figure work with Table A. An attempt to do so, however, shows that the method is not so simple as this, but that it is necessary to perform the three operations described by the author in his explanation.

Suppose, for example, it is required to calculate $\log 5$.

The tables give $0{\cdot}698 = \log 4{\cdot}998,884,875$, so that 0·698 are the first three figures of $\log 5$ with a remainder in the antilogarithm of 0·011,155,125. If we use the $\log a$ method *either* with Table A *or* with Table B there is not much difficulty in obtaining the next three figures 970, but the table does not contain sufficient data to find the final remainder required to give the last three figures. The only way is to use the tables *backwards* in order to obtain the antilogarithm of 0·698,970 correct to 9 decimals. This involves a second use of the tables with the assistance of the $\log a$ column, and leads to the result

$$\log 4{\cdot}999,999,954 = 0{\cdot}698,970.$$

The final remainder is 0·000,000,046, from which a repetition of the first operation gives the last three figures of the logarithm, namely 004.

It will thus be seen that the calculation of logarithms to 9 figures is three times as difficult as their calculation to 6 figures. It involves three distinct operations instead of one, and the author is therefore justified in calling the tables A and B respectively single and triple extension tables. It is not often that logarithms are actually required to 9 places, and the emergency may be provided for by keeping a copy of the tables and learning to work Table B when necessity arises, rather than by keeping a big cumbersome book of tables which is never to be found when it is wanted, and still involves cumbersome work in the matter of interpolation.

It should be pointed out that the class teacher who sends the copies of the explanatory notice to the bookbinder to have the pages cut in order to prevent his pupils from wasting their time, will find the pockets for holding the tables cut off. French people are as a rule mortally afraid of the guillotine, but unless this is used the pages should be made larger than the pockets, which they are not.

G. H. BRYAN.

Einführung in die Theorie der Differentialgleichungen mit einer unabhängigen Variabeln. Von DR. LUDWIG SCHLESINGER. (*Sammlung Schubert*, XIII.) (Leipzig, 1904.)

Gewöhnliche Differentialgleichungen beliebiger Ordnung. Von DR. J. HORN. (*Sammlung Schubert*, L.) (Leipzig, 1905.)

The subject of differential equations has grown to such an extent that it is necessary for the writer of a text-book to confine himself to a particular aspect of the subject. The works that we have before us deal with what might be called the formal side of the subject, that is the considerations connected with the existence of solutions, the nature of the integrals in the vicinity of the singularities, and the form of the equations when the singularities are of a specified type.

Dr. Schlesinger's treatise is a good introduction to the higher work on the subject. The writer, aiming at simplicity, considers only equations of the first two orders, a feature which will be welcome to many readers since it enables them to obtain a grip of the essential ideas without having to master any difficult analysis.

It is a significant fact that the differential equations which arise out of problems in geometry and physics are no other than the equations

with the simplest types of singularities; this point is brought out clearly in Schlesinger's treatise, Riccati's equation for instance being obtained when the differential equation

$$\frac{dy}{dx}=f(x, y)$$

is supposed not to possess an integral with a variable branch point.

Another good feature which might be noticed is that in obtaining the definite integral solutions of the various equations the writer calls attention to the fact that in each case the method depends upon the construction of a relation of the form

$$D_x\{u(x, z)\}=\Delta_z\{v(x, z)\},$$

and we believe that it will be through the development of the theory of relations of this type that further progress in this branch of the subject may be expected.

Dr. Horn's aim has been to write a more elaborate treatise on the subject avoiding as much as possible a repetition of the portions treated by Schlesinger. The characteristics of the work are the frequent occurrence of systems of linear differential equations and the use that has been made of the canonical forms of a linear substitution.

In dealing with a system of ordinary linear differential equations with constant coefficients

$$\frac{dy_r}{dx}+\sum_{s=1}^{n}a_{rs}y_s=f_r(x) \quad (r=1, \dots n)$$

the writer employs Weierstrass' method which depends upon the reduction of the system to the canonical form

$$\frac{dY_1}{dx}=rY_1+F_1(x),$$

$$\frac{dY_2}{dx}=rY_2+Y_1+F_2(x),$$

$$\frac{dY_n}{dx}=rY_n+Y_{n-1}+F_{n-1}(x).$$

This process is more scientific than the usual one since it avoids the use of operators and is in keeping with the general plan of the work; we do not think, however, that it is quite so convenient in actual practice as the symbolic method.

A good feature of the book is that it gives an account of much recent work that has been done on the subject; thus, besides the valuable chapters on the asymptotic representation of an integral and infinite determinants, we find that mention is made of Painlevé's canonical forms of differential equations of the second order and of a certain type whose integrals are uniform functions.

The writer also does well to call attention to theorems that are related to some of the recent mathematical work, as for instance the theorem that if the coefficients $P_i(x, \mu)$ $(i=1, \dots n)$ of the linear differential equation

$$\frac{d^ny}{dx^n}+P_1(x, \mu)\frac{d^{n-1}y}{dx^{n-1}}+\dots P_n(x, \mu)y=0$$

are continuous functions of the real variable x in the interval $|x-a|\leqq r$ and whole rational functions of the complex parameter μ. A solution y which is such that the initial values of $y, y' \ldots y^{(n-1)}$ for $x=a$ are independent of μ is a whole transcendental function of μ for values of x within the given range (see p. 297), a theorem which corresponds to an important result in the theory of integral equations.

The printing is good in both cases and I have not noticed any misprints. H. BATEMAN.

MATHEMATICAL NOTES.

205. [L'. 2. b.]. [A solution, not by elliptic functions is wanted of the following:

Given five lines *abcde* in a plane, it is known that the pairs of points *ab*, *ce*; *bc*, *da*; *cd*, *eb*; *de*, *ac*; *ea*, *bd* are in a collineation. Prove that the fixed triangle of this collineation is self polar as to both the conic on (*ab*, *bc*, *cd*, *de*, *ea*) and the conic on (*ac*, *ce*, *eb*, *bd*, *da*). (F. MORLEY).]

Solution by H. BATEMAN.

Lemma. Let a collineation referred to its fixed triangle be given by the equations

$$x'=ax, \quad y'=by, \quad z'=cz,$$

then the conic

$$S\equiv Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$$

will correspond to

$$S\equiv Aa^2x^2+Bb^2y^2+Cc^2z^2+2Fbcyz+2Gcazx+2Habxy=0,$$

and a point $P(\xi, \eta, \zeta)$ will correspond to the point $P'\left(\frac{\xi}{a}, \frac{\eta}{b}, \frac{\zeta}{c}\right)$ on the one hand and to $P(a\xi, b\eta, c\zeta)$ on the other.

The polar of P' with regard to S is

$$Aax\xi+Bby\eta+Ccz\zeta+F(c\eta z+b\zeta y)+G(c\xi z+a\zeta x)+H(a\eta x+b\xi y)=0,$$

and the polar of P with regard to S' is

$$Aax\xi+Bby\eta+Ccz\zeta+F(b\eta z+c\zeta y)+G(a\xi z+c\zeta x)+H(b\eta x+a\xi y)=0.$$

These lines will be the same if

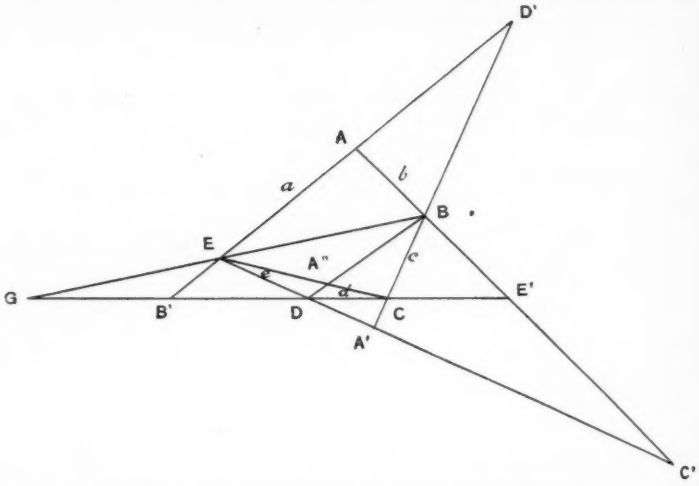
$$\frac{Aa\xi+Ha\eta+Ga\zeta}{Aa\xi+Hb\eta+Gc\zeta}=\frac{Hb\xi+Bb\eta+Fb\zeta}{Ha\xi+Bb\eta+Fc\zeta}=\frac{Gc\xi+Fc\eta+Cc\zeta}{Ga\xi+Fb\eta+Cc\zeta}=\lambda, \text{ (say).}$$

Eliminating ξ, η, ζ we get a cubic for λ, hence there will in general be three positions of P for which the lines are the same; if there are more than three positions of P, we must have $F=G=H=0$, and then the conics S and S' will have the fixed triangle as a self-conjugate triangle, and the two lines will be the same for all positions of P.

To apply this lemma to the theorem in question, let $ABCDE$ be the pentagon formed by the five lines *abcde*, $A'B'C'D'E'$ the pentagon formed by the alternate sides. We can show at once that A, A'; B, B'; C, C'; D, D'; E, E' are in a collineation, by comparing the cross-ratios of the pencils $A\{BCDE\}$, $A'\{B'C'D'E'\}$, and the corresponding pencils for the other corners.

Let S be the conic $ABCDE$, S' the conic $A'B'C'D'E'$. Now the two points which correspond to A in the collineation are A'' on the one hand (for EC corresponds to $E'C'$ and BD to $B'D'$) and A' on the other; also the polar of A'' with regard to S is easily seen to be the line $A'G$, and by considering

the Pascal hexagon $A'D'B'E'C'A'$ we see that this line is the tangent at A' to the conic S'; accordingly the polar of A'' with regard to S is the same line as the polar of A' with regard to S'. Taking the five points $ABCDE$



in turn, we have five pairs of points for which this condition is satisfied; hence from the above lemma we conclude that the conics S and S' must have the fixed triangle of the collineation as a self-conjugate triangle.

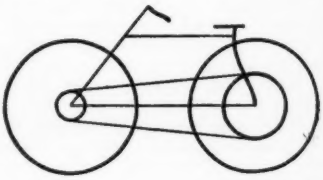
206. [D. 2. a.]. From Prof. Elliott's remark, I gather that he *presupposes* Dedekind's "Schnitt" theory. Of course, if this is done, and thus real numbers are defined (as 'Schnitte'), the sufficiency of the criterion of convergence can be proved, and this was done by Dedekind himself, at the end of his *Stetigkeit und irrationale Fahlen*, more shortly and, I think, more simply than in Prof. Elliott's note.

But I thought that Prof. Elliott was attempting to prove the sufficiency of the criterion *without* any previous arithmetical theory of irrationals (since he did not mention any), and I pointed out the impossibility of this.

PHILIP E. B. JOURDAIN.

207. [X. 4.]. "*Graphs.*"—If in many examinations you set a question on tracing a simple conic, say, $x^2-y^2+x+y-1=0$, you are ruled out because "General Equation of the Second Degree is not in the syllabus." The remedy is to replace the simple conic by a difficult *cubic* curve, the question then becomes a "graph" and is accepted without opposition. G. H. BRYAN.

208. [X. 4.]. *Figure of a Bicycle.*—The annexed figure or something like it was drawn by a candidate in an examination in Applied Mathematics in a question on Rigid Dynamics. The writer *believes* that that candidate passed. G. H. BRYAN.

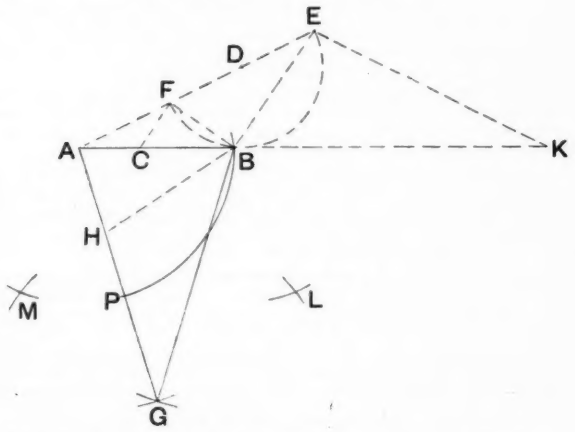


209. [V. a.]. Mr. Trachtenberg (195 [X. 1]) suggests the use of $\frac{292}{89}$ feet as the value of a metre correct to 5 places of decimals. Unfortunately, though it seems mere scholastic pedantry to mention it, this value is founded on the old legal value of a metre as 39·37079 inches and not on the new value 39·37011 inches. Many text-books and examination papers fall into the same error. F. W. DOBBS.

210. [K. 9.] I. *Construction of a regular pentagon.*

Mr. A. F. Van der Heyden asks for a simple line of argument with regard to the construction of a pentagon. Wishing to keep the construction, and being unwilling to give much space to it, I worked out the following method which is given in substance in pp. 71, 77, 78 of my "Geometry." It is as simple as any in construction, simpler than most in argument; it is entirely geometrical, and moreover affords excellent practice in similar triangles.

If AB is divided at C so that CB is the mean proportional of AB, AC, then I have called CB the *mean part* of AB (this is clearly a better name than *medial*) and the triangle AGB of Eucl. IV. 10 I have called the *mean base triangle*. AB is also said to be divided *in mean* at C.



If AB is any line, perp. $BD=\frac{1}{2}AB$, $AFDE$ drawn to meet the circle centre D, radius DB; then $FE=2DB=AB$.

Also angle $ABF=AEB$ in opposite arc, and triangle AFB ||| * triangle ABE.

$$\therefore \frac{AF}{FE}=\frac{AF}{AB}=\frac{AB}{AE}=\frac{FE}{AE}.$$

$\therefore$ FE or AB is the mean part of AE.

Draw $FC \parallel BE$; $\therefore$ CB is the mean part of AB (similar division).

We have constructed both CB the mean part of AB, *and* AE *of which* AB *is the mean part.*

Make $AG=BG=AE$; then ABG is the mean base triangle, which I have shown has angle $A=B=2G$.

It is easier however to suppose $A=B=72°=2G$.

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* ||| "is similar to."

Draw BH bisecting angle B; $\therefore$ triangle HBG is isosceles, and triangle $ABH \parallel\!\!\parallel BAG$; (angles are 72°, 36°, 72°)

$$\therefore\ HG = HB = AB.$$

Also since BH bisects angle B in triangle ABG

$$\frac{AH}{HG} = \frac{AB}{BG} = \frac{HG}{AG};$$

$\therefore$ HG, *i.e.* AB, is the mean part of AG, and $\therefore$ $AG = AE$.

To construct pentagon: If AB is radius of circle, construct triangle ABG (this is very quickly done), and arc BP, centre A; then BP is the side of the in-pentagon.

If the side AB is given, construct triangle ABG; and construct vertices L, M by arcs centres B, G, A, radius AB.

Then figure $BHGL$ is a rhombus;

$$\therefore \text{ angle } BLG = BHG = 108° = (\text{similarly})\ AMG,$$

$$\text{angle } ABL = 3 \times 36° = 108° = (\text{similarly})\ BAM \text{ and } MGL;$$

$$\therefore\ ABLGM \text{ is a regular pentagon.}$$

The construction is quickly written out. 'Construct AE of which AB is the mean part, construct isosceles triangle ABG, having $AG = BG = AE$, etc.' Or 'Construct the mean base triangle ABG, base AB, etc.'

The line EK parallel to FB determines the external point K such that BK is the mean proportional of AB, AK. This is distinctly better than the circle construction from centre A, radius AE, since there is no chance of getting the wrong point, and no proof is required.

Another feature entirely wanting in the traditional construction is that by continuing the series of parallels to EB, FB from C and K, we determine an endless series of mean parts (since the successive triangles vertex A are similar).

Thus
$$\begin{aligned} AC &= \text{mean part of } AF \\ &= \text{mean part of mean part of } AB \\ &= (\text{mean part})^3 \text{ of } AE \\ &= (\text{mean part})^4 \text{ of } AK, \text{ etc.} \end{aligned}$$

It is at once seen also that if CB is the mean part of AB, then AC is the mean part of CB; or if b is the mean part of a, then $a - b$ is the mean part of b, and $2b - a$ the mean part of $a - b$, $2a - 3b$ of $2b - a$ and so on continually.

For AF is the mean part of AB and $= CB$,
and AC „ „ AF *i.e.*, of CB. E. BUDDEN.

211. [**V. a.**] *Shortened Method of Multiplication, etc.*

I most heartily endorse Mr. Godfrey's objection to these, particularly in multiplication. The shortened method of multiplication is not only longer, but from the nature of the case more inaccurate, as it cannot be tested by the rule of nines; and mere reworking may easily repeat a mistake once made. The stopping in division when enough figures are reached is obvious and is sufficient. A slight fraction of time is saved in taking square roots to 5 figures (which are often wanted to get 3 correct in quadratics), by getting the last two by division. This is, I am sure, all that is worth while teaching; moreover it can easily be taught as the original rule, and there is no difficulty with the decimal point. Whenever I have to teach the shortened method in multiplication I always think of the new shorthand clerk who was taking down notes in longhand, and when asked if he could not write shorthand, said 'Yes, but it takes me longer to write than the other.'

E. BUDDEN.

COLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

Wanted.

Mathematical Gazette. 1st Quarto Number. T. J. Garstang, M.A., Bedales School, Petersfield, Hants.

BOOKS, ETC., RECEIVED.

A School Geometry. Parts I. and II. *With Introductory Course of Experimental and Practical Work.* By H. S. Hall and F. H. Stevens. Pp. viii, 138, iv. 2s. 6d. 1906. (Macmillan.)

The Theory of Sets of Points. By W. H. Young and Mrs. Grace Chisholm Young. Pp. xii, 316. 12s. net. 1906. (Cambridge University Press.)

A Second Geometry Book. By J. G. Hamilton and F. Kettle. Pp. viii, 292. 3s. 6d. 1906. (Arnold.)

Mathematical Drawing. By Prof. G. M. Minchin and Prof. J. B. Dale. Pp. viii, 145. 7s. 6d. net. 1906. (Arnold.)

Text-book of Mechanics. By Prof. L. A. Martin, Jr. Vol I. Statics. Pp. xii, 142. 1906. (Wiley, Chapman and Hall.)

Über die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert. By M. Simon. Pp. viii, 278. 8 m. 1906. (Teubner.)

Zur Ausgleichung der Widersprüche in den Winkelbedingungsgleichungen Trigonometrischer Netze. By L. Krüger. Pp. 34. 2m. 80. 1906. (Teubner.)

Die Absolute Wahrheit der Euclidischen Geometrie. By A. Lippmann. Pp. 68. 1906. (Gerstäcker, Leipzig.)

Leçons sur les Séries Trigonométriques. By H. Lebesgue. Pp. 128. 3 fr. 50. 1906. (Gauthier-Villars.)

A System of Applied Optics, being a complete system of the second order, and the foundation of a complete system of the third order. With examples of their applications. By H. Dennis Taylor. Pp. xvi, 334. 30s. net. 1906. (Macmillan.)

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